How the Risk Measures play important roles for Tail Risk Management and Diversification

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Abstract

In the world of investment, the subject of building a portfolio concerning tail risk is still one of the frequently discussed subjects and unquestionably vital for investors. This paper seeks to examine how the risk measures, lower tail-dependence based on the copulas approach and Conditional Value-at-Risk (CVaR), affect the portfolio strategies and play important roles for tail risk management and diversify the portfolio. By using these two risk measures mentioned above, two different types of risk-based portfolios are proposed that consider for the tail risks: 1) Minimum-lower tail-dependence portfolio (RMTP) and 2) Risk Parity Portfolio based on Conditional Value-at-Risk (CRPP).

The simulation results showed how those two risk-based portfolios, RMTP and CRPP, work effectively in multi-asset allocation framework with six assets: equities and sovereign bonds of Japan, United States and Germany, based on the monthly rebalance rule, using 2004-2018 sample period. One of the key findings were that both RMTP and CRPP strategies delivered better performances compared with the traditional portfolio strategies in terms of sharp ratio: 1) RMTP yielded 0.92 and 2) CRPP yielded 0.99 (by adding an appropriate risk reduction to this portfolio, the sharp ratio went up to 1.76). In addition, both of these two strategies also worked effectively in terms of the average of maximum monthly drawdown related to the effect of the tail risk: 1) RMTP by 1.80% and 2) CRPP by 1.74% (by adding an appropriate risk reduction to this portfolio, maximum drawdown decreased to 0.78%).

Furthermore, this paper also studies an enhancement strategy based on Risk Parity Portfolios (RPP) focusing on and using co-integration relationship (co-integration approach). According to the simulation result, this proposed enhancement strategy has a potential to yield roughly 4.5% return. Finally, this paper presents the explicit derivation of lower tail-dependence and co-integration approach.

Keywords: Risk Parity, Copulas approach, Lower Tail-Dependence, Hamilton Jacobi Bellman Equation, Co-integration relationship, Conditional Value-at-Risk.
Introduction

Since the collapse of Lehman Brothers in 2008, some practitioners have begun to claim that another huge financial crisis is approaching, even though nobody seems to know what could trigger it. Under this kind of environment, asset owners may become more nervous of the risks emerging from their own portfolios. Recently, the correlation and dependence among the global markets, especially between the stocks and bonds, have become stronger than before. Therefore the variety of ways to manage portfolios that focus on the correlation and dependence are gaining more attention. Markowitz (1952) is one of the pioneers in portfolio optimization. He proposed the mean-variance framework. Markowitz’s idea is to minimize risk exposure by diversifying, taking advantages of covariance effects which contribute to risk reduction in portfolios. His framework is designed to minimize risks having given a constant level of expected portfolio returns. Of course, this optimization problem can also be formulated as maximizing expected portfolio returns based on a fixed degree of risk. The solutions by solving the optimization problems are the same. What is important here is that, his idea is based on both expected returns and risks. On the contrary, Risk Parity Portfolio (RPP), proposed by Qian (2005, 2006), is based only on risks\(^1\). Qian’s approach is formulated and assumes that each asset contributes an equal amount of risk to portfolios. It is obvious that the main difference between these two approaches is whether expected returns are taken into consideration or not.

Here, let us think about expected return and risk respectively. First of all, what is the expected return? It is the level of future return that asset owners expect. Moreover, expected return can be generated and realized only when asset owners take risks. If the expected return is realized within the target range, in other words, if the portfolio manager has a confident outlook, mean-variance may be one of the better suited approaches to construct portfolios. However in the real world of investment, expected returns cannot always be realized just simply by taking risks. Conversely, it is often said that estimation of risk is not as difficult as estimation of expected return. This is because of the well-known clustering effect in the volatility.

Secondly, what is risk? Should it be calculated by the whole risks which include both the downside risks (lower tail risk) and the upside risks (upper tail risk)? If we assume a symmetric joint distribution to measure risks, the assessment will be unprecise since lower side and upper side risks are different. It is vital to point out that especially under the situation

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1 Although this paper assumes Risk Parity Portfolio such as Qian’s approach, in recent research, there is a new approach for Risk Parity, which is based on the both historical returns and their variance in the construction of an optimal, diversified investment portfolio (Akhilesh and Joel 2018).
where the next financial crisis is approaching; it is wiser to pay more attention to lower tail risk rather than upper tail risk in portfolio selection.

Furthermore, it is critical to think more intensely about risks especially in asset allocation framework that consist equities and bonds. This is because; it is classically assumed that equities and bonds have negative correlation. This means that a fall in stock value will be offset by an increase in bond value to some extent. Needless to say, the strength of the correlation is time varying, by all means. It is easy to imagine of a situation where there is a positive correlation and both assets yield positive returns, which is of course, good for asset owners. On the contrary, the worst case scenario is having a positive correlation and both assets yield negative returns. It is unfortunate for asset owners of the fact that, it is difficult to estimate the worst scenarios, which may damage their own portfolios. This is caused by the fact that, there are not enough samples available to discuss statistically. It is quite obvious that enough sample size is crucial for statistical evidence in order to even start such discussion.

Moreover, what if a worst case scenario is estimated with the whole return data, where lower tail, upper tail and middle parts are mixed in return distribution? Risks may not be calculated precisely if the tail risk is not being considered, due to the existing sample size in lower tail being smaller than the other parts. It is quite clear that, risks will be skewed to the domain with more samples. In normal situation, we often hear a phrase of something being “beyond our expectation”. The same thing can be applied to the subject of tail risk. The author believes that “beyond our expectation” could happen due to these reasons mentioned above, not measuring the risk precisely, and not paying attention to lower tail. Nevertheless, it is also true that paying too much attention to risks may lead to losing opportunities of gaining positive returns. Therefore, it is recommendable for asset owners to duly face risks.

There are a numerous approaches to defend portfolios from tail risk. For example: Stop-loss strategies, Option based portfolio insurance (Brennan and Schwarz (1976), Leland (1980)), Constant proportion portfolio insurance (Black and Jones (1987)), Ratcheting Strategies, and Value-at-Risk based portfolio insurance (see the Thomas et al (2014)), as shown in Dangl (2014). This paper examines and focuses on the following two risks: Lower tail-dependence based on copulas approach and Conditional Value-at-Risk (CVaR).

1) Lower tail-dependence based on copulas:
Copulas are famous approaches as multivariate modeling tools. Copulas are methods for associating random variables together, irrespective of their marginal distributions. Copulas are multivariate distributions whose marginal distributions are all uniform distribution over (0,1). The main purpose of copulas are to separate marginal distribution from correlation and this is done by transforming each variable to distribute in a uniform. When the

“Lower tail-dependence and CVaR are expected to be the effective risk measures to estimate lower tail risk in comparison with standard deviation based.”
specification of the dependence of multivariate random variables is of interest, this aspect can be looked at further with the theory of copulas. Copulas consider the dependence structure between random variables independently. By applying these copulas, tail dependence coefficients can be calculated as the measure of the interactive strength between underlying assets on both upper tail risk and lower tail risk.

2) Conditional Value-at-Risk: CVaR:
Value-at-Risk (VaR) is well known as one of the risk measure for tail risks, which is recommended by the Basel Committee. An upper percentile of the loss distribution is called Value-at-Risk. Its popularity comes from a simple and easy to understand representation of high losses. VaR may be effective in the situation where the underlying assets are normally distributed. However, for non-normal distributions, VaR may have undesirable properties. A number of literatures claim and criticize that it is not a convex measure of risks, and does not allow us to solve the optimization problem containing VaR measure, for instance. Adding to this, VaR gives a percentile of loss distribution that does not provide an adequate domain of the possible losses in the tail of the distribution (Iakovos and Berc 2013). On the other hand, CVaR clears those two main problems. Regarding the former one, Rockafellar and Uryasev (2000, 2002) proposed a minimization formulation, in which CVaR is built in, as convex or linear optimization problem. This is also a risk measure closely related to VaR, and be defined as the conditional expected loss under the condition. Regarding the latter one, CVaR is defined as the conditional expected loss under the distribution above VaR. Therefore, CVaR risk measure allows us to measure the fat-tailed risks.

This paper seeks to examine empirical study of how these two risk measures, lower tail-dependence based on the copulas approach and CVaR are effective in comparison with the normal risk measure, which is standard deviation based. These two risk measures can be applied to Minimum Variance Portfolio (MV) and RPP in the multi-assets framework containing equities and bonds of Japan, United States and Germany, six assets in total.

To simplify, two portfolios are created. One portfolio is risk-based portfolio of minimum tail-dependence (RMTP), and the other is RPP with CVaR. RMTP is formulated as the linear programming problem to minimize the lower tail-dependence matrices, which are similar to MV formation. Yet, in RMTP, lower tail-dependence matrices are used instead of covariance matrices. It is constructed with the lower tail-dependence coefficients of each pairs, which is two combinations of a set of six and its diagonal elements are 1. To calculate the lower tail-dependence coefficients, 15 copulas are prepared as candidates. The criterion of selection of copulas is based on Akaike Information Criteria (AIC): Gaussian copula, t-copula, Frank copula, Gumbel copula, Clayton copula

"Using these two risk measures, this paper suggests two portfolios:
1) Minimum lower tail-dependence portfolio (RMTP) and
2) Risk Parity Portfolio based on Conditional Value-at-Risk (CRPP).
Furthermore, this paper also studies an enhancement strategy based on CRPP focusing and using co-integration relationship."
and Joe copula as candidates. 90 degrees, 180 degrees and 270 degrees rotated for Gumbel, Clayton and Joe, whereas Risk Parity Portfolio based on CVaR is formulated as the normal Risk Parity such as Qian (2005, 2006), except the risk measure. CVaR is used instead of standard deviation.

These two risk measures and the earlier suggested portfolios in this paper are expected to realize the optimal weight of each asset class, considering the lower tail risk, which are to be compared with the MV and RPP. Moreover, as an application of RPP, this paper proposes an enhancement strategy based on co-integration relationship between stocks and bonds. There are some previous research that state there is a co-integration relationship between equity and bonds (Niall 2002). Note that these synthetic securities of equity and bond, which are constructed based on RPP respectively, are not genuine as underlying assets.

Undoubtedly, there has already been a substantial amount of studies done on the copulas approach and CVaR, yet, only few studies have attempted to apply those ideas of which this paper suggests, to asset allocation framework empirically. This paper will contribute to empirical analysis using above framework and to present the explicit derivation of lower tail-dependence and co-integration approach.

The rest of the paper is presented as follows. Section 2 describes the two formulations of the optimization problem. Section 3 explains the empirical analysis using the formulation in section 2. Section 4 describes how to enhance the RPP. Finally, Conclusion.

**Risk-Based Portfolio Considering Tail Risk**

This section shows two types of risk-based portfolios that concern for tail risk. 2.1 describes minimum tail-dependence portfolio with the copulas approach. 2.2 explains how to calculate the lower tail-dependence based on the copulas approach. 2.3 illustrates Risk Parity, in which the risk is calculated by CVaR instead of normal standard deviation.

### 2.1 Minimum-lower tail-dependence portfolio

This subsection outlines, firstly, MV with N’s assets which is formulated by

\[
\text{arg min}_w \quad \frac{1}{2} w^T \Sigma w
\]

\[s.t. \quad \sum_{i} w_i = 1\]

\[w_i \geq 0\]  

(2.1)

Where

- \( w \) : weight vector of n’s assets.
- \( w_i \) : weight of i’s asset.
- \( \Sigma \) : variance covariance matrices of \( n \times n \).
As above shows, solution from (2.1) is based on risks of each asset. Contrarily to this, minimum tail-dependence portfolio is formulated by

\[
\begin{align*}
\arg\min_{\mathbf{w}} & \quad \frac{1}{2} \mathbf{w}^T \mathbf{T} \mathbf{w} \\
\text{s.t.} & \quad \sum_{i} w_i = 1 \\
& \quad w_i \geq 0
\end{align*}
\]

(2.2)

Where

- \( \mathbf{w} \): weight vector of n’s assets.
- \( w_i \): weight of i’s asset.
- \( \mathbf{T} \): lower tail-dependence matrices of \( n \times n \) calculated by the copulas approach.

The risk measure in formulation of (2.2) differs from (2.1). The risk of (2.2) is lower tail-dependence based on the copulas approach. As regards to the calculation of lower tail-dependence, it is portrayed in the next subsection. Further retransformation of (2.2), RMTP can be formulated as below.

\[ \xi = D^{1/2} \mathbf{w} \]

Where

- \( \xi \): the second intermediate vector of asset weights.
- \( D \): the diagonal matrix of asset variance at time \( t \) with \( \sigma_i^2 \) at its \( (i,i) \) elements and zero on all off-diagonal elements.

In addition, rescale the second intermediate asset weight vector of the total weight in order to transform the sum of the final weight as below.

\[ w_i = \frac{\xi_i}{\sum_{j=1}^{N} \xi_j} \]  

(2.3)

Note: In this paper, (2.3) is used for the RMTP construction which will be used in the Section 3.

2.2 Copula selection and Calculus for lower tail-dependence coefficients

This subsection demonstrates how to calculate the lower tail-dependence coefficients. As regards to the mathematical description of the process to calculate the lower tail-dependence coefficients, see Appendices 1.
Here, the author will mention why the copula\(^2\) approach is superior. Copulas provide greater flexibility in modelling the multivariate distribution by allowing us to fit the appropriate marginal to different random variables and then to specify the appropriate copula function that bind these marginal distributions together. However, traditional representations of multivariate distributions require that all random variables have the same marginal distribution. Since a copula can capture dependence structures regardless of the form of the margins, a copula approach to modelling related variables is potentially significantly useful in risk management. These advantages imply that copulas provide a superior approach to the modelling of multivariate statistical problems.

Based on the copulas such as above description, the author selects the best fitted copula among all copula candidates including Gaussian copula, t-copula, Frank copula, Gumbel copula, Clayton copula and Joe copula as candidates\(^3\). Moreover, Gumbel, Clayton and Joe, those that are rotated by 90 degrees, 180 degrees and 270 degrees are also candidates. Thus, the total number of candidates is 15 copulas including rotated ones. The selection criterion is based on AIC.

First of all, taking previous researches into account, the author chooses Skew-t distribution for the marginal distribution estimation. In comparison with other distributions, Skew-t distribution\(^4\) is generally known as one of the flexible distribution to capture the return characteristic for any assets. To parameterize Skew-t distribution, the author maximizes the log likelihood with multi-phased using EM algorithm\(^5\). After parameterization of the coefficients of each asset’s marginal distribution respectively, transforms it to uniform distribution. Thereafter maximize the following log likelihood equation to find the best fitted copula among the candidates. Suppose two dimensional copula, and each random variables have N’s dataset. \(x_i^j\) denotes the j data of number i random variable. Let \(F(\cdot; \Psi_1)\) is a marginal distribution of random variable i, with parameter \(\Psi_1\), let \(c(\cdot; a)\) is a copula density function with parameter \(a\).

Then,

\[
\ln (\Psi_{X_1}, \Psi_{X_2}; a) = \ln \left( \prod_{j=1}^{N} \left( c \left( \left( F_{X_1}(x_{1}^j; \Psi_{X_1}), F_{X_2}(x_{2}^j; \Psi_{X_2}) \right); a \right) \prod_{i=1}^{N} f_i(x_{i}^j; \Psi_i) \right) \right) \\
= \sum_{j=1}^{N} \ln c \left( \left( F_{X_1}(x_{1}^j; \Psi_{X_1}), F_{X_2}(x_{2}^j; \Psi_{X_2}) \right); a \right) + \ln \left( \prod_{i=1}^{N} \prod_{j=1}^{N} f_i(x_{i}^j; \Psi_i) \right) \\
= \sum_{j=1}^{N} \ln c \left( \left( F_{X_1}(x_{1}^j; \Psi_{X_1}), F_{X_2}(x_{2}^j; \Psi_{X_2}) \right); a \right) + \sum_{j=1}^{N} \sum_{i=1}^{N} \ln \left( f_i(x_{i}^j; \Psi_i) \right) \quad (2.4).
\]

\(^2\) For more theoretical detail on copulas, see Nelson (2006) as an introduction of copulas.

\(^3\) In this paper, the methodology to estimate the copulas is based on IMF as to follow the previous research.

\(^4\) Skew-t probability density function: \(f(x; \xi, \omega, a, v) = 2\omega^{-1} f(\sqrt{\omega^2 z^2 + \xi^2}) \left( 1 + \frac{z^2}{v} \right)^{-\frac{v+1}{2}} (\omega^{-1} f(z; v)T(z; v) + 1) (-\infty < x < \infty)\) where \(z = \frac{x - \xi}{\omega}, \tau = \left( \frac{v + 1}{v} \right)^{\frac{1}{2}} f(z; v) \left( \frac{1 + \frac{z^2}{v}}{\gamma} \right)^{\frac{v+1}{2}} (-\infty < z < \infty), T(z; v) = \int_{-\infty}^{\frac{z}{\sqrt{\gamma}}} t(u; v) du.\)

\(^5\) For more details, see Schefller (2008).
Finally, lower tail-dependence can be calculated based on the best fitted copula of two assets in pairs. Lower tail-dependence matrices of $n \times n$ T can be made by transformation. By following the process of the calculus for lower tail-dependence coefficients in Appendices 1, the solution can be found in Table 1.

**Table 1: Tail-dependence coefficients for each copula**

<table>
<thead>
<tr>
<th>Copulas</th>
<th>$\lambda_u$</th>
<th>$\lambda_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>t</td>
<td>$\frac{1}{3} - \frac{\sqrt{3}}{3} \left( \frac{1 - \rho^2 \frac{1}{3} + \frac{1}{3}}{\left(1 + \rho^2 \frac{1}{3} + \frac{1}{3} \right)^{1/2}} \right)$</td>
<td>$\frac{1}{3} - \frac{\sqrt{3}}{3} \left( \frac{1 - \rho^2 (\frac{1}{3} + \frac{1}{3})}{\left(1 + \rho^2 \frac{1}{3} + \frac{1}{3} \right)^{1/2}} \right)$</td>
</tr>
<tr>
<td>Clayton</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>Frank</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Joe</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>180 degree rotated Clayton</td>
<td>$2^{1/2}$</td>
<td>0</td>
</tr>
<tr>
<td>180 degree rotated Gumbel</td>
<td>$2 - 2^{1/2}$</td>
<td>0</td>
</tr>
<tr>
<td>180 degree rotated Joe</td>
<td>0</td>
<td>$2 - 2^{1/2}$</td>
</tr>
<tr>
<td>90 degree rotated Clayton</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90 degree rotated Gumbel</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90 degree rotated Joe</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>270 degree rotated Clayton</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>270 degree rotated Gumbel</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>270 degree rotated Joe</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: $\lambda_u$ denotes upper tail-dependence coefficient, $\lambda_L$ denotes lower tail-dependence coefficient.

*(Source: Nissay Asset Management)*

### 2.3 Risk Parity Portfolio with Conditional Value-at-Risk (CRPP)

Differ from the previous subsection, this subsection describes the other type of portfolio construction, using CVaR measure based on the Risk Parity Portfolio (CRPP). Firstly, let’s see the process of constructing RPP. Using $r_i$ and $w_i$ to denote the return and weight for each individual asset $i$, the portfolio’s return and standard deviation can be written as

$$ r_p = \sum_{i=1}^{N} w_i r_i $$

$$ \sigma_p = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}} $$

where $\sigma_{ij}$ is the covariance between assets $i$ and $j$, and $\sigma_{ii} = \sigma_i^2$ is the variance of asset $i$. 
Let’s see two measures of risk contributions that will be useful for defining the understanding RPP. The first one is the marginal risk contribution:

$$MRC_i = \frac{\partial \sigma_p}{\partial w_i} = \sum_{j=1}^{N} w_j \cdot \sigma_{ij} = \text{cov}(r_i, r_p)$$

The above tells us the impact on an infinitesimal increase in an asset’s weight on the total portfolio risk, measured by the standard deviation.

The second one is the total risk contribution.

$$TRC_i = w_i \frac{\partial \sigma_p}{\partial w_i} = \sum_{j=1}^{N} w_i \cdot w_j \cdot \sigma_{ij} = w_i \cdot \text{cov}(r_i, r_p)$$

This gives us a way to break down the total risk of the portfolio into separate components. To see why this is the case, notice that

$$\sum_{i=1}^{N} TRC_i = \sigma_p^2$$

$$w_i \frac{\partial \sigma_p}{\partial w_i} = w_i \cdot \frac{\partial \sigma_p}{\partial w_j} = \lambda \text{ } \forall i, j \neq i$$ (2.5)

where $\lambda$ is the constant to be found.

To find the optimal Risk Parity weights solution, this paper follows the Maillard, Roncalli and teiletch (2010)

$$\sum_{i=1}^{N} \sum_{j=1}^{N} (TRC_i - TRC_j)^2$$

subject to $\sum_{i=1}^{N} x_i = 1$.

$R(w)$ is a risk measure. In this paper, $R(w)$ is calculated by CVaR $\mathbb{E}[|X| - \text{VaR}_\alpha(X)]$, where $\text{VaR}_\alpha(X) = -\inf\{x | P(X \leq x) > \alpha\}$, differs from the normal RPP measured by standard deviation.

**Empirical Analysis**

**Data Description**

The Dataset includes daily prices of three equity futures and three sovereign bond futures: Japanese government bond future (JB), US government bond future (TY), German government bond future (RX), TOPIX (Japanese equity) future (TP), SP (US equity) future (SP), DAX (German equity) future (GX) from Bloomberg. The sample period starts from April 2003 and ends in December 2018. Table 2 and Table 3 provide descriptive statistics of annualized return, annualized volatility, sharp ratio and correlations over the entire sample period.
Table 2: Basic statistics of each asset

<table>
<thead>
<tr>
<th>JB</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
<th>TY</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
<th>RX</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>3.94%</td>
<td>3.04%</td>
<td>1.30</td>
<td>−1.25%</td>
<td>5.04%</td>
<td>−0.20</td>
<td>5.38%</td>
<td>4.00%</td>
<td>1.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2005</td>
<td>−4.12%</td>
<td>3.31%</td>
<td>−1.25</td>
<td>−2.46%</td>
<td>4.33%</td>
<td>−0.57</td>
<td>−1.09%</td>
<td>3.92%</td>
<td>−0.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2006</td>
<td>0.56%</td>
<td>3.59%</td>
<td>0.16</td>
<td>1.62%</td>
<td>3.65%</td>
<td>0.44</td>
<td>−1.80%</td>
<td>3.61%</td>
<td>−0.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2007</td>
<td>4.51%</td>
<td>3.74%</td>
<td>1.21</td>
<td>9.36%</td>
<td>6.54%</td>
<td>1.43</td>
<td>1.00%</td>
<td>4.91%</td>
<td>0.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2008</td>
<td>−1.48%</td>
<td>5.49%</td>
<td>−0.27</td>
<td>4.62%</td>
<td>10.74%</td>
<td>0.43</td>
<td>7.02%</td>
<td>7.53%</td>
<td>0.93</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2009</td>
<td>0.09%</td>
<td>2.72%</td>
<td>0.03</td>
<td>−5.98%</td>
<td>7.18%</td>
<td>−0.83</td>
<td>−0.68%</td>
<td>5.56%</td>
<td>−0.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2010</td>
<td>0.96%</td>
<td>3.01%</td>
<td>0.32</td>
<td>2.48%</td>
<td>6.60%</td>
<td>0.38</td>
<td>−1.43%</td>
<td>6.16%</td>
<td>−0.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2011</td>
<td>1.89%</td>
<td>1.99%</td>
<td>0.65</td>
<td>8.24%</td>
<td>5.97%</td>
<td>1.38</td>
<td>13.03%</td>
<td>7.92%</td>
<td>1.65</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2012</td>
<td>2.32%</td>
<td>1.65%</td>
<td>1.41</td>
<td>1.92%</td>
<td>4.07%</td>
<td>0.47</td>
<td>4.96%</td>
<td>6.54%</td>
<td>0.76</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2013</td>
<td>−0.51%</td>
<td>3.04%</td>
<td>−0.17</td>
<td>−6.21%</td>
<td>5.45%</td>
<td>−1.16</td>
<td>−1.28%</td>
<td>4.90%</td>
<td>−0.26</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2014</td>
<td>1.72%</td>
<td>1.71%</td>
<td>1.01</td>
<td>4.20%</td>
<td>4.36%</td>
<td>0.96</td>
<td>9.86%</td>
<td>4.35%</td>
<td>2.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2015</td>
<td>2.63%</td>
<td>2.21%</td>
<td>1.19</td>
<td>1.23%</td>
<td>5.17%</td>
<td>0.24</td>
<td>2.93%</td>
<td>6.78%</td>
<td>0.43</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2016</td>
<td>−0.63%</td>
<td>1.96%</td>
<td>−0.32</td>
<td>−4.29%</td>
<td>4.35%</td>
<td>−0.99</td>
<td>−0.95%</td>
<td>5.88%</td>
<td>−0.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2017</td>
<td>0.32%</td>
<td>1.05%</td>
<td>0.30</td>
<td>−2.62%</td>
<td>3.38%</td>
<td>−0.78</td>
<td>−1.09%</td>
<td>4.58%</td>
<td>−0.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2018</td>
<td>1.48%</td>
<td>1.17%</td>
<td>1.27</td>
<td>0.87%</td>
<td>3.53%</td>
<td>0.25</td>
<td>4.79%</td>
<td>4.20%</td>
<td>1.14</td>
<td></td>
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</tr>
</tbody>
</table>

Ave. 0.84% 2.88% 0.29 0.82% 5.70% 0.14 2.65% 5.59% 0.47

<table>
<thead>
<tr>
<th>TP</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
<th>SP</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
<th>GX</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>4.90%</td>
<td>11.74%</td>
<td>0.05</td>
<td>2.38%</td>
<td>7.99%</td>
<td>10.06%</td>
<td>0.79</td>
<td>13.22%</td>
<td>12.05%</td>
<td>1.10</td>
<td></td>
</tr>
<tr>
<td>2005</td>
<td>37.76%</td>
<td>15.90%</td>
<td>2.38</td>
<td>9.69%</td>
<td>9.82%</td>
<td>0.99</td>
<td>31.32%</td>
<td>12.20%</td>
<td>2.57</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2006</td>
<td>0.94%</td>
<td>17.59%</td>
<td>0.05</td>
<td>9.57%</td>
<td>10.60%</td>
<td>0.90</td>
<td>15.59%</td>
<td>15.57%</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2007</td>
<td>−29.68%</td>
<td>25.84%</td>
<td>−1.15</td>
<td>−5.78%</td>
<td>18.34%</td>
<td>−0.31</td>
<td>−3.25%</td>
<td>20.08%</td>
<td>−0.16</td>
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</tr>
<tr>
<td>2008</td>
<td>−32.18%</td>
<td>46.35%</td>
<td>−0.69</td>
<td>−39.19%</td>
<td>44.03%</td>
<td>−0.89</td>
<td>−37.60%</td>
<td>39.79%</td>
<td>−0.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2009</td>
<td>23.86%</td>
<td>19.29%</td>
<td>1.24</td>
<td>38.29%</td>
<td>17.97%</td>
<td>2.13</td>
<td>40.94%</td>
<td>21.26%</td>
<td>1.93</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2010</td>
<td>−9.26%</td>
<td>21.49%</td>
<td>−0.43</td>
<td>13.59%</td>
<td>17.68%</td>
<td>0.77</td>
<td>15.13%</td>
<td>17.49%</td>
<td>0.87</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2011</td>
<td>0.40%</td>
<td>16.74%</td>
<td>0.02</td>
<td>8.30%</td>
<td>22.39%</td>
<td>0.37</td>
<td>2.38%</td>
<td>28.19%</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2012</td>
<td>19.97%</td>
<td>17.27%</td>
<td>1.16</td>
<td>11.23%</td>
<td>13.23%</td>
<td>0.85</td>
<td>12.42%</td>
<td>17.11%</td>
<td>0.73</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2013</td>
<td>17.59%</td>
<td>26.42%</td>
<td>0.67</td>
<td>17.56%</td>
<td>11.40%</td>
<td>1.54</td>
<td>20.87%</td>
<td>14.91%</td>
<td>1.40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2014</td>
<td>25.27%</td>
<td>16.65%</td>
<td>1.52</td>
<td>10.27%</td>
<td>11.74%</td>
<td>0.88</td>
<td>23.01%</td>
<td>16.78%</td>
<td>1.37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2015</td>
<td>−9.47%</td>
<td>26.40%</td>
<td>−0.38</td>
<td>0.89%</td>
<td>16.25%</td>
<td>0.05</td>
<td>−14.38%</td>
<td>24.95%</td>
<td>−0.58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2016</td>
<td>13.20%</td>
<td>20.59%</td>
<td>0.64</td>
<td>13.90%</td>
<td>10.05%</td>
<td>1.38</td>
<td>21.43%</td>
<td>16.48%</td>
<td>1.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2017</td>
<td>13.03%</td>
<td>13.09%</td>
<td>1.00</td>
<td>11.58%</td>
<td>11.41%</td>
<td>1.01</td>
<td>−0.83%</td>
<td>12.81%</td>
<td>−0.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2018</td>
<td>−7.67%</td>
<td>17.53%</td>
<td>−0.44</td>
<td>6.66%</td>
<td>15.98%</td>
<td>0.42</td>
<td>−6.35%</td>
<td>15.94%</td>
<td>−0.40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ave. 4.71% 22.60% 0.21 7.63% 18.26% 0.42 8.97% 20.50% 0.44

Note: Mean (annualized return), Sd (annualized volatility), SR (sharp ratio), Ave (average for each statistics) over the whole period.

(Source: Nissay Asset Management)

As Table 2 shows, there is a variation across assets in terms of returns and volatilities. TY has the lowest return at 0.82%, whereas the best performers are DAX (8.97%) and SP (7.63%). Relatively volatile asset classes are TOPIX (22.60%), and DAX (20.50%). This dataset is interesting because of these differences and potentials for diversification as seen in correlations across asset classes shown in Table 3 below.
Table 3: Correlation matrices between each asset

<table>
<thead>
<tr>
<th></th>
<th>JB</th>
<th>TY</th>
<th>RX</th>
<th>TP</th>
<th>SP</th>
<th>GX</th>
</tr>
</thead>
<tbody>
<tr>
<td>JB</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TY</td>
<td>0.10</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RX</td>
<td>0.15</td>
<td>0.52</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP</td>
<td>-0.37</td>
<td>-0.08</td>
<td>-0.11</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SP</td>
<td>-0.06</td>
<td>-0.33</td>
<td>-0.26</td>
<td>0.15</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>GX</td>
<td>-0.11</td>
<td>-0.32</td>
<td>-0.37</td>
<td>0.29</td>
<td>0.61</td>
<td>1.00</td>
</tr>
</tbody>
</table>

(Source: Nissay Asset Management)

Simulation Results

Firstly, the author mentions about the RMTP. As a first step, check the uniformity of each marginal distribution. Testing by Kolmogorov-Smirnov test, all marginal distributions cannot be rejected by the null hypothesis, which is skew-t follows uniform distribution, with 5% level of significance. Using the formulation on the Table 1 and the calculation with coefficients derived from the parameterization, RMTP weight can be calculated. Table 4 shows the result of RMTP and MV. This simulation is based on the performance of monthly rebalance, Non-forward looking strategies. The dataset is daily frequency. The covariance matrix and lower tail-dependence coefficients are estimated using the previous one year data.

Table 4: Summary of MV and RMTP simulation

<table>
<thead>
<tr>
<th>Year</th>
<th>Mean MV</th>
<th>Std MV</th>
<th>SR MV</th>
<th>Mean RMTP</th>
<th>Std RMTP</th>
<th>SR RMTP</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>3.96%</td>
<td>2.47%</td>
<td>1.61</td>
<td>4.71%</td>
<td>2.50%</td>
<td>1.88</td>
</tr>
<tr>
<td>2005</td>
<td>-0.93%</td>
<td>2.38%</td>
<td>-0.29</td>
<td>3.23%</td>
<td>2.34%</td>
<td>1.42</td>
</tr>
<tr>
<td>2006</td>
<td>0.42%</td>
<td>2.49%</td>
<td>0.17</td>
<td>1.52%</td>
<td>2.64%</td>
<td>0.50</td>
</tr>
<tr>
<td>2007</td>
<td>2.86%</td>
<td>2.73%</td>
<td>1.05</td>
<td>1.77%</td>
<td>2.43%</td>
<td>0.73</td>
</tr>
<tr>
<td>2008</td>
<td>-0.95%</td>
<td>4.74%</td>
<td>-0.22</td>
<td>-3.94%</td>
<td>6.08%</td>
<td>-0.45</td>
</tr>
<tr>
<td>2009</td>
<td>0.13%</td>
<td>2.45%</td>
<td>0.05</td>
<td>2.38%</td>
<td>2.52%</td>
<td>0.94</td>
</tr>
<tr>
<td>2010</td>
<td>1.23%</td>
<td>2.26%</td>
<td>0.03</td>
<td>1.01%</td>
<td>2.49%</td>
<td>0.41</td>
</tr>
<tr>
<td>2011</td>
<td>0.81%</td>
<td>1.93%</td>
<td>0.04</td>
<td>3.25%</td>
<td>2.19%</td>
<td>1.25</td>
</tr>
<tr>
<td>2012</td>
<td>2.82%</td>
<td>1.35%</td>
<td>2.16</td>
<td>5.09%</td>
<td>1.59%</td>
<td>3.21</td>
</tr>
<tr>
<td>2013</td>
<td>-0.12%</td>
<td>2.03%</td>
<td>-0.40</td>
<td>1.80%</td>
<td>3.20%</td>
<td>0.55</td>
</tr>
<tr>
<td>2014</td>
<td>3.10%</td>
<td>1.56%</td>
<td>1.99</td>
<td>5.83%</td>
<td>1.88%</td>
<td>2.13</td>
</tr>
<tr>
<td>2015</td>
<td>2.20%</td>
<td>2.25%</td>
<td>1.00</td>
<td>1.02%</td>
<td>2.63%</td>
<td>0.36</td>
</tr>
<tr>
<td>2016</td>
<td>-1.08%</td>
<td>1.94%</td>
<td>-0.52</td>
<td>0.29%</td>
<td>2.09%</td>
<td>0.14</td>
</tr>
<tr>
<td>2017</td>
<td>0.83%</td>
<td>1.08%</td>
<td>0.59</td>
<td>1.75%</td>
<td>1.81%</td>
<td>0.92</td>
</tr>
<tr>
<td>2018</td>
<td>-0.98%</td>
<td>1.13%</td>
<td>-0.67</td>
<td>-1.54%</td>
<td>1.51%</td>
<td>-1.02</td>
</tr>
<tr>
<td>Ave.</td>
<td>1.08%</td>
<td>2.21%</td>
<td>0.59</td>
<td>1.89%</td>
<td>2.58%</td>
<td>0.92</td>
</tr>
</tbody>
</table>

(Source: Nissay Asset Management)

As Table 4 shows, MV is heavily concentrated in the JB with a weight of roughly 60% over the last 15 years. Whereas, equities have small allocations, the range is 0 to 4% weight. On the contrary to it, RMTP is efficiently allocated to every assets; JB weight is 44%. This result can be said that the risk diversification is achieved. The RMTP has a relatively larger exposure to the more volatile asset classes (equities) and a relatively small exposure to less volatile ones (bonds). This result implies that standard deviation may carry a tendency to overestimate the genuine tail risk. This may be a natural result because standard deviation is being calculated by the whole return distribution. **In terms of sharp ratio, Table 4 shows that RMTP outperforms MV by roughly 0.3 points.**
As the author mentioned above, overestimating leads to a loss in the opportunity to gain the returns. As a result, in terms of sharp ratio, RMTP outperforms MV by roughly 0.3 points.

Secondly, the author mentions about the CRPP. Table 5 shows the result of the simulation of CRPP, compared with RPP. At a first glance, the difference between CRPP and RPP may not be understood to a large extent in terms of total performance and portfolio weight. However, Table 6 implies the performance difference when financial crisis occur. Table 6 is the summary of the maximum drawdown (MAXDD) from 2004 to 2018. In this paper MAXDD is calculated on the basis of monthly rebalance (let $DD_t$ denotes drawdown of portfolio at monthly period $t$. Ave in Table 6 describes the average of $DD_t$ and Max describes the maximum of $DD_t$ in the observation period). As shown in Table 6, the average of MAXDD of RPP is 1.83%. On the contrary to this, CRPP is 1.74%, being lower 9bps than RPP is. In terms of the maximum of MAXDD, there is over 50bps difference. This implies that CVaR measure contributes to the tail risk management as it is expected. Most of the base portfolio rebalance frequency is one month more or less in the rule based or quantitative investment framework. Thus, if the market crushes in the middle of the month, investors cannot change their own portfolio weights, generally speaking. What is more, their portfolio may be wounded. In such circumstance, it is natural to come up with the idea of another strategy that is not monthly monitoring but daily monitoring, such as setting up a daily stopping time.
Table 5: Summary of RPP and CRPP simulation

<table>
<thead>
<tr>
<th>Year</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
<th>Mean</th>
<th>Sd</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>4.59%</td>
<td>2.53%</td>
<td>1.81</td>
<td>4.69%</td>
<td>2.55%</td>
<td>1.80</td>
<td>4.09%</td>
<td>2.07%</td>
<td>1.98</td>
</tr>
<tr>
<td>2005</td>
<td>4.18%</td>
<td>2.47%</td>
<td>1.69</td>
<td>4.65%</td>
<td>2.57%</td>
<td>1.81</td>
<td>4.66%</td>
<td>2.10%</td>
<td>2.23</td>
</tr>
<tr>
<td>2006</td>
<td>2.02%</td>
<td>2.84%</td>
<td>0.71</td>
<td>1.86%</td>
<td>2.89%</td>
<td>0.64</td>
<td>3.11%</td>
<td>2.43%</td>
<td>1.28</td>
</tr>
<tr>
<td>2007</td>
<td>1.49%</td>
<td>2.48%</td>
<td>0.66</td>
<td>1.86%</td>
<td>2.53%</td>
<td>0.74</td>
<td>1.62%</td>
<td>1.94%</td>
<td>0.84</td>
</tr>
<tr>
<td>2008</td>
<td>-4.42%</td>
<td>5.90%</td>
<td>-0.75</td>
<td>-4.12%</td>
<td>5.63%</td>
<td>-0.73</td>
<td>-0.18%</td>
<td>4.43%</td>
<td>-0.04</td>
</tr>
</tbody>
</table>

The difference between CRPP and RPP may not be understood to a large extent in terms of total performance and portfolio weight. However, as shown in Table 6, in terms of MAXDD, there is over 50bps difference between CRPP and RPP. This result proves the effectiveness of the proposed risk measures.

Table 6: Maximum drawdown of each portfolio (%)}

<table>
<thead>
<tr>
<th>Year</th>
<th>MV</th>
<th>RMTP</th>
<th>RPP</th>
<th>CRPP</th>
<th>CRPP with appropriate risk control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ave</td>
<td>2.13</td>
<td>1.80</td>
<td>1.83</td>
<td>1.74</td>
<td>0.78</td>
</tr>
<tr>
<td>Max</td>
<td>7.21</td>
<td>10.16</td>
<td>10.58</td>
<td>10.03</td>
<td>6.62</td>
</tr>
</tbody>
</table>

(Source: Nissay Asset Management)
Here, add the risk reduction model to CRPP. Let the risk reduced function be $F_{\text{risk reduction}}(\text{Market condition}, \text{direction})$ which takes three types of value: 1, 0.25 and 0. This function is made based on the principle market philosophy. It is true that there are a variety of philosophies about market. Once you go to a book store, you can find a large amount of technical books on how to beat the market. However, the author believes that the genuine essence of market philosophy does not exist so many, as people may think. One of the essences to win the market is to know the market condition, whether the market is in the trend or the range. It is easy to imagine how important the market condition is. If the investor thinks the market is in the trend, just simply bet on the trend. This strategy often called “Buy and Hold”. On the contrary, if the investor thinks the market is in the range market, just buy at the near bottom and sell at the ceiling repeatedly. This is sometimes called “Revolving” or “Range trade”. Therefore, to know the market condition is the key, which is input in the risk reduced function F as one of the parameters. Yet it is not enough to give the value as signal. This is because even if the market condition is in the trend, how do we judge whether trend is downward or upward? For example, take a look at Figure 1. Do you think this figure shows the market is in the trend? This is a visual trick of the human eyes. Most people will likely say that the market shown in Figure 1 will keep going up.

**Figure 1: A trick of the eyes**

![Figure 1](source: Nissay Asset Management)

However, for machine or model based quantitative approach, this judgement as to whether the market is in the trend or range, sometimes becomes difficult and occasionally misjudged. This is because as shown in Figure 2, if we measure the trend with period 1, the trend is to go down.
If measured with period 2, the trend is to go up. To clarify, it is important to
decide the length of the window to measure the future trend. The author
believes that the length of the window should not be fixed but should be
flexible, considering the pattern of ticks or structure of times series. Based on
these two factors (Market condition, direction), the risk reduction
function F gives a three type signals. In this paper, further detailed
description about this function F is not mentioned but the author points out
that F is made by appropriate risk control. Further detail, it shall be written in
the author’s next coming paper.

CRPP with appropriate risk control is equipped with this risk control model,
and it is noticeably superior to the base portfolio CRPP in terms of sharp
ratio. As written above, the risk control model delivers three numerical
values as signal. This contributes to the portfolio as the risk control. For
instance, if the value is 0.25, the investment weight is reduced as parallel.

As shown in the Figure 3, the domain of red in the downside is smaller than the
blue one. This implies that downside risk is well controlled and protected
from losing money. Some may claim that this is natural because of the risk
reduction function F. The maximum value from F is 1, whereas the minimum
value is 0. However the smaller domain of red is not just because of these
values generated from the function F. The domain of red in the upside is not
so small that the upside opportunity maybe missed. In other words, this
appropriate model is not too passive toward risk-return, but just simply
controls effectively. As another result, the average of MAXDD of CRPP
with risk appropriate control is less than 1%. The MAXDD is the smallest
among the model. In addition, Figure 4 and Figure 5 show the wealth process
of each portfolio and time-series weight vector of CRPP.
Figure 3: Monthly return comparison between CRPP and risk-controlled CRPP

(Source: Nissay Asset Management)

Figure 4: Wealth Process of each portfolio

(Source: Nissay Asset Management)

Figure 5: Time series of weight vector of CRPP

(Source: Nissay Asset Management)
Enhancement CRPP with Co-integration approach

Up to this point, two types of portfolio have been introduced. In this section, the author introduces the way to enhance CRPP focusing on the property of co-integration relationship between CRPP of stocks (CRPP-S) and CRPP of bonds (CRPP-B). CRPP-S is the RPP that consists of strictly stocks only, and the same applies to CRPP-B. In practice, some portfolio managers often build CRPP-S and CRPP-B respectively to obtain clear market outlook from the standpoint of macroeconomics or politics and so on.

This section introduces the one of the ideas how to enhance the CRPP strategy using with the characteristics of co-integration between CRPP-S and CRPP-B. Thanks to Johansen (1998), Engle and Granger (1987) and Johansen (1992), the author uses (n,m)- Error Correction model ( (n,m)-EC model) ; n is the number of asset and m is the number of co-integration relationship. Suppose, N’s assets $S_t = (S_{1,t}, ..., S_{n,t})$ can be represented in the non-stationary vector autoregressive processes. Then, with Granger representation theorem, co-integration relationship can be formulated as follows.

$$
\Delta S_t = \sum_{i=1}^{n-1} \xi_i \Delta S_{t-1} + \alpha - BA' S_{t-1} + \epsilon_t
$$

Where
- $A$ and $B : n \times m$ matrices.
- $A' S_{t-1} \sim I(0)$ represents m’s co-integration relationship.
- $A'$ : a co-integration vector.

Note: Co-integration rank is determined by the rank of $-BA'$, which takes up to n-1. $-BA' S_{t-1}$ is called error correction term.

Based on this co-integration vector, pairs trading strategy can be built, which is born in Wall Street practitioners\(^6\).

In the following subsection, the model is represented with generalized form. If you want to know more concrete and explicit derivation, see the Appendices 2.

\(^6\) As famous references, see Gatev et al (2006), Elliot et al (2006) to understand these primitive strategies, which are sometimes called pairs trading strategy. Tourin and Yan (2013) is the first literature describing the dynamic pairs trading strategy in the multi-period framework, with the co-integration term built in the drift term. They also provide the closed form solution in two types of assets.
4.1 Model
This subsection describes how to derive the closed form solution for simulation in subsection 4.2. Suppose (n-m)-EC model, the investment horizon is finite, $T < \infty$. Risky assets in co-integration relationship $S_t = (S_{1,t}, \ldots, S_{n,t})$ follow the stochastic differential equation (SDE)

$$d \log S_{1t} = \left(\mu_i + \beta_i'z_t - \frac{1}{2}\Sigma_s\right)dt + GLdB_t \quad (4.1).$$

Where
- $I = 1, \ldots, n$.
- $G = \text{diag}(\sigma_{s_1}, \ldots, \sigma_{s_n})$.
- $L$: lower triangle matrices derived from Cholesky decomposition of correlation matrices R.
- $\Sigma_s = GRG'$.
- $B_t = (B_{i,t})$ is n-dimensional Brownian motion defined on filtered probability space $(\Omega, F, P)$.
- $z_{1t} = y_{10} + \sum_{i=1}^{n} y_{1i} \ln S_{1t}, 1 = 1, \ldots, m$.

Note that $y_{10}$ and $y_{1i}$ are $a$ and $A'$ in (4.1). To simplify, risk free rates is set as 0 here. $\pi_t = (\pi_{1,t})$ is the optimal holding ratio of risky asset $i$ at time $t$, based on no-self-financing condition, the author defines the wealth process.

$$dW_t = \sum_{i=1}^{n} \pi_{1t}dS_{1t} \quad (4.2).$$

To define the value function with paying attention to that representative individual faces the maximization problem for (4.2) at terminal point.

$$u(t, w, s) = \sup_{\pi_t \in A_t} E_t\left[U(W_T), W_t = w, S_t = s\right] \quad (4.3).$$

Note that (4.3) satisfies the condition of integrable property $E\left[\int_t^T (\pi_u S_u)^2 du\right] < \infty$.

$A_t$ is a set of investment at time $t$. Utility function for the derivation of closed form solution is set exponential type in this paper.

$$U(w) = -e^{-\lambda w} \quad (4.4).$$

Where
- $\lambda$: risk aversion coefficient, which takes $\lambda > 0$.

Combine (4.1), (4.2), (4.3) and (4.4), the problem is set as follow.
\[
\sup_{\pi_t \in \mathcal{A}_t} E_t \left[ U(W_t) \right] \\
\text{s.t. } d \log S_{tt} = \left( \mu_t + \beta_t z_t - \frac{1}{2} \Sigma_t \right) dt + \mathbf{G} \mathbf{L} d\mathbf{B}_t \\
dW_t = \sum_{i=1}^n \pi_{ti} dS_{it} \\
U(w) = -e^{-\lambda w}.
\]

Thanks to Bellman Dynamic Programming principle, using with Taylor expansion and Ito formula, Hamilton Jacobi Bellman (HJB) Equation can be derived as following (4.5), and it can be treated as static programming problem instead of dynamic programming.\(^7\)

\[
u_t + \sup_{\pi_t \in \mathcal{A}_t} [S\pi_t(\mu + \beta z)u_w + (\mu + \beta z)Su_s + S\pi_t \Sigma_t u_{ww} + \frac{1}{2} S\pi_t \Sigma_t \pi_t u_{ww} + \frac{1}{2} S\Sigma_t \pi_t S_{uu}] = 0
\] (4.5)

Let’s notate terminal condition \(u(T, w, s) = U(w), \ u_w = \partial u(\cdot, w, \cdot) / \partial w\). In here, let \(x_i = \log S_{it}, \ i = 1, \ldots, n\), use Ansatz for \(u(t, w, x_1, \ldots, x_n) = U(w)g(t, x_1, \ldots, x_n)\), after that use Call-Hop transformation\(^8\) to eliminate the non-linearity, finally optimal trading strategy for every period as following can be derived in the closed form solution.

\[
\pi_t S_t = \frac{1}{\lambda} \left( (G^G)^{-1} (\mu + \beta z) + \gamma (-2a(t)z - b(t)) \right) \quad (4.6).
\]

Where
- \(i = 1, \ldots, n\).
- \(a(t) = \frac{1}{2} \beta' (GG')^{-1} \mathbf{b}(T - t)\).
- \(b(t) = \mu' (GG')^{-1} \beta(T - t) + \left( -\frac{1}{2} ((GG')' \gamma) \right) \beta'(GG')^{-1} \beta (T - t)^2 / 2\).

4.2 Simulation

The author used the variance and covariance estimates computed from the entire sample to construct the back test portfolios. Note that, since forward-looking information was used for computing portfolio weights, the strategies are not real-time investable, differ from the simulation in Section 3. The sample period is Jan 2010 - Aug 2013. There is a co-integration relationship by ADF test at 5% level (CRPP-S and CRPP-B are unit root process). For this dataset and for a risk tolerance \(\gamma = 0.1\), as Figure 6 illustrates, this strategy can yield roughly 4.5% profit.

\(^7\) For more detail about the process of derivation, see the appendices 2.1.
\(^8\) For more detail about the process of derivation, see the appendices 2.2 - 2.4.
Conclusion

This paper seeks to examine empirical study of how CVaR and lower tail-dependence are effective to diversify the portfolio and profitable portfolio as a result of diversification. This paper also explored the explicit derivation of lower tail-dependence and co-integration approach. In non-forward looking simulation in terms of sharp ratio, RMTP yielded 0.92, CRPP yielded 0.99, and CRPP with appropriate risk control yielded 1.76. In terms of the tail risk management, RMTP protected its maximum drawdown within the 1.80%, CRPP 1.74%, CRPP with appropriate risk control yielded 0.78% also. These values were superior against the benchmark portfolio where the normal standard deviation was used. Therefore, it can be concluded that using risk measures that pay more attention to tail risk are more effective in multi-asset framework. In addition, the enhancement strategy focusing on the co-integration relationship yielded roughly 4.5% returns. These results also prove the effectiveness of the suggested two risk measures in this paper.
1. Calculation of tail dependence matrices

1.1 Calculation of tail dependence

Thanks to Sklar theorem\(^9\), two dimensional copula \(C\) has a property as below.

\[
F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) = C \left( F_{X_1}(x_1), F_{X_2}(x_2) \right)
\]

\[
C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))
\]

Where, \(F_{X_1, X_2}(x_1, x_2)\) is the simultaneous distribution function of random variables \(X_1, X_2\), \(F_{X_1}(x_1)\), \(F_{X_2}(x_2)\) are the marginal distribution of \(X_1, X_2\).

Due to the above theorem, one can treat simultaneous distribution function composed of the different marginal distribution even there is a correlation. On top of that, if copula function can be differentiated,

\[
f(x_1, x_2) = c(F_1(x_1), F_2(x_2)) \prod_{i=1}^2 f_i(x_i) \quad (A. 1.1)
\]

Now, let’s define a copula \(C(u_1, u_2)\) of two risks of \(X_1, X_2\). Simultaneous distribution function \(C(u_1, u_2)\) of uniform random vectors \(U_1, U_2\), upper tail-dependence coefficients \(\lambda_U\) and lower tail-dependence coefficients \(\lambda_L\) of \(X_1, X_2\) can be calculated by the following.

\[
\lambda_U \triangleq \lim_{u \to 1^-} P(X_2 > F_2^{-1}(u)|X_1 > F_1^{-1}(u)) = \lim_{u \to 1^-} \frac{1 - P(X_1 \leq F_1^{-1}(u)) - P(X_2 \leq F_2^{-1}(u)) + P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))}{1 - P(X_1 \leq F_1^{-1}(u))}
\]

\[
= \lim_{u \to 1^-} \left( 1 - \frac{1 - u}{1 - u} \right)
\]

\[
= \lim_{u \to 1^-} \left( 1 - 1 \frac{\partial C(s, t)}{\partial s} \bigg|_{s=t=u} + 1 \frac{\partial C(s, t)}{\partial t} \bigg|_{s=t=u} \right)
\]

\[
= \lim_{u \to 1^-} \left( P(U_2 > u | U_1 = u) + P(U_1 > u | U_2 = u) \right).
\]

\[
\lambda_L \triangleq \lim_{u \to 0^+} P(X_2 < F_2^{-1}(u)|X_1 < F_1^{-1}(u)) = \lim_{u \to 0^+} \frac{C(u, u)}{u}.
\]

\(^9\) For more specific and theoretical essence about copula, see Nelsen [1998].
1.2 Candidate of Copula

Copulas used in this paper are as follows.

The Gaussian copula has the same structure as the multidimensional Gaussian distribution. It is given by

\[ C(u_1, u_2; \Sigma) = \Phi_{\Sigma}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)) \]

Where \( \Sigma \) is correlation matrices.

The t copula has the same structure as the multidimensional t distribution. It is given by

\[ C(u_1, u_2; \Sigma, \nu) = t_{\nu}(t_{\nu}^{-1}(u_1), t_{\nu}^{-1}(u_2)) \]

Where \( \Sigma \) is correlation matrices and \( \nu \) is the degree of freedom.

Clayton Copula

The Clayton copula is an asymmetric Archimedean copula. This copula has characteristics that lower tail-dependence is greater than the upper tail-dependence. This copula is given by

\[ C(u_1, u_2) = \left( \sum_{i=1}^{2} u_i^{-\theta} - 1 \right)^{-\frac{1}{\theta}} \]

Where \( \theta > 0 \).

Gumbel Copula

The Gumbel copula is an asymmetric Archimedean copula. This copula has characteristics that upper tail-dependence is greater than the lower tail-dependence. This copula is given by

\[ C(u_1, u_2) = \exp \left( - \left[ (-\ln(u_1))^{\theta} + (-\ln(u_2))^{\theta} \right]^{\theta^{-1}} \right) \]

Where \( \theta > 1 \).

Frank Copula

The Frank copula is a symmetric Archimedean copula. This copula is given by

\[ C(u_1, u_2) = -\frac{1}{\theta} \ln \left( 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \right) \]

Where \( \theta > 0 \).
**Joe Copula**

The Joe copula is given by

\[ C(u_1, u_2) = 1 - \left( 1 - (1 - u_1)^\theta (1 - (1 - u_2)^\theta) \right)^\frac{1}{\theta}. \]

Note: 90 degree routed, 180 degree routed and 270 degree routed can be calculated as follows.

90 degree rotated: \( \tilde{C}(u_1, u_2) = u_2 - C(1 - u_1, u_2) \).
180 degree rotated: \( \tilde{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2) \).
270 degree rotated: \( \tilde{C}(u_1, u_2) = u_1 - C(u_1, 1 - u_2) \).

1.3 Example of the derivation lower tail-dependence

This subsection shows the examples of how to calculate the lower tail-dependence. Firstly, take 90 degree routed Joe as an example.

90 degree rotated Joe copula is given by

\[ C(u_1, u_2) = u_2 - C(1 - u_1, u_2) = \left( 1 - \left( (u_1^\theta + (1 - u_2)^\theta - u_2^\theta (1 - u_2)^\theta) \right)^\frac{1}{\theta} \right). \]

Follow the Appendices 1.1, tail-dependence can be calculated below.

\[ \lambda_U = \lim_{t \to 1-} \frac{1 - 2t + C(t, t)}{1 - t} = \lim_{t \to 1-} \frac{-t + (t^\theta + (1 - t)^\theta - t\theta(1 - t)^\theta)^\frac{1}{\theta}}{1 - t} \]
\[ = \lim_{t \to 1-} \left[ 1 - \left\{ t^\theta + (1 - t)^\theta - t\theta(1 - t)^\theta \right\}^\frac{1}{\theta} \left\{ t^{\theta-1} - (1 - t)^{\theta-1} - t^{\theta-1} \right\} \right] \]
\[ = 0. \]

\[ \lambda_L = \lim_{t \to 0+} \frac{C(t, t)}{t} = \lim_{t \to 0+} \frac{t - \left( 1 - (t^\theta + (1 - t)^\theta - t\theta(1 - t)^\theta)^\frac{1}{\theta} \right)}{t} \]
\[ = \lim_{t \to 0+} \left[ 1 + \frac{1}{\theta} \left\{ t^\theta + (1 - t)^\theta - t\theta(1 - t)^\theta \right\}^\frac{1}{\theta} \left\{ t^{\theta-1} - \theta t^{\theta-1} - \theta t^{\theta-1} \right\} \right] \]
\[ = \lim_{t \to 0+} \left[ 1 + \left\{ t^\theta + (1 - t)^\theta - t^\theta(1 - t)^\theta \right\}^\frac{1}{\theta} \left\{ t^{\theta-1}(1 - t)^{\theta-1} \right\} \right] \]
\[ = 0. \]

Other copulas also can be derived as same as the above way except 90 degree routed Gumbel copula. This can be calculated as follows.
\( C(u_1, u_2) = u_2 - C(1 - u_1, u_2) = u_2 - \exp\left\{ -\left( (-(\ln(1 - u_1))^\gamma + (-\ln u_2)^\gamma \right)^{\frac{1}{\gamma}} \right\} \)

\[
\lambda_u = \lim_{t \to 1^-} \frac{1 - 2t + C(t, t)}{1 - t} = \lim_{t \to 1^-} \frac{1 - 2t}{1 - t} - \exp\left\{ -\left( (-(\ln(1 - u_1))^\gamma + (-\ln u_2)^\gamma \right)^{\frac{1}{\gamma}} \right\}
\]

Then, let \( t \) be \( t \in (1 - \varepsilon, 1) \),

\[
\max((-\ln(1 - t))^\gamma, (-\ln t)^\gamma) \leq \{( -\ln(1 - t))^\gamma + (-\ln t)^\gamma \}^{\frac{1}{\gamma}}
\]

Also

\[
-\ln(1 - t) \leq \max((-\ln(1 - t), -\ln t) - \ln t \leq \max(-\ln(1 - t), -\ln t)
\]

Therefore

\[
\{( -\ln(1 - t))^\gamma + (-\ln t)^\gamma \}^{\frac{1}{\gamma}} = \{( -\ln(1 - t)(-\ln(1 - t))^\gamma - (-\ln t)(-\ln t)^\gamma \}^{\frac{1}{\gamma}}
\]

\[
\leq \{( -\ln(1 - t))(\max(-\ln(1 - t), -\ln t))^\gamma - (-\ln t)(\max(-\ln(1 - t), -\ln t))^\gamma \}^{\frac{1}{\gamma}}
\]

\[
= \{( -\ln(1 - t) - \ln t)(\max(-\ln(1 - t), -\ln t))^\gamma \}^{\frac{1}{\gamma}}
\]

\[
= (-\ln(1 - t) - \ln t)^\frac{1}{\gamma}
\]

\[
= -\ln(1 - t) - \ln t.
\]

Therefore,

\[
\exp\{(-\ln(1 - t) - \ln t)\} \leq \exp\left\{ -\{( -\ln(1 - t))^\gamma + (-\ln t)^\gamma \}^{\frac{1}{\gamma}} \right\} \leq \exp\{(-\ln(1 - t))\}
\]

\[
\frac{1 - t - \exp\left\{ -\{( -\ln(1 - t))^\gamma + (-\ln t)^\gamma \}^{\frac{1}{\gamma}} \right\}}{1 - t} \leq \frac{1 - t - \exp\{(-\ln(1 - t))\}}{1 - t}
\]

\[
\lim_{t \to 1^-} \frac{1 - t - \exp\{(-\ln(1 - t) - \ln t)\}}{1 - t} = \lim_{t \to 1^-} \frac{1 - t - t(1 - t)}{1 - t} = 0
\]

\[
\lim_{t \to 1^-} \frac{1 - t - \exp\{(-\ln(1 - t))\}}{1 - t} = \lim_{t \to 1^-} \frac{1 - t - (1 - t)}{1 - t} = 0
\]

Using with squeeze theorem,

\[
\lambda_u = \lim_{t \to 1^-} \frac{1 - t - \exp\{ -\{( -\ln(1 - t))^\gamma + (-\ln t)^\gamma \}^{\frac{1}{\gamma}} \}}{1 - t} = 0
\]

As same approach as \( \lambda_u \) by let \( t \) be \( t \in (0, 0 + \varepsilon) \), \( \lambda_L \) can be checked

\[
\lambda_L = \lim_{t \to 0^+} \frac{t - \exp\{ -\{( -\ln(1 - t))^\gamma + (-\ln t)^\gamma \}^{\frac{1}{\gamma}} \}}{t} = 0.
\]
2. Calculation of the weight

2.1 Derivation of HJB Equation

This subsection shows the process of derivation of HJB equation below.

\[ u(t, W_t, S_t) = \sup_{(\pi^1, \pi^2, \pi^3) \in \Lambda_{t+dt}} E_t w_t S_t \left[ \int_t^{t+dt} e^{-r(s-t)} U(s, W_s, S_s, (\pi^1_s, \pi^2_s, \pi^3_s)) ds \right. \]

\[ \left. \quad + \sup_{(\pi^1, \pi^2, \pi^3) \in \Lambda_{t+dt}} E_t w_t W_t + dW_t S_t + ds_t \right] \]

\[ \left. \int_t^{t+dt} e^{-r(s-t)} U(s, W_s, S_s, (\pi^1_s, \pi^2_s, \pi^3_s)) ds \right] \tag{A.1} \]

Second term on the right side of (A.2.1) can be rewritten in the following.

\[ \sup_{(\pi^1, \pi^2, \pi^3) \in \Lambda_{t+dt}} E_t w_t W_t + dW_t S_t + ds_t \left[ \int_t^{t+dt} e^{-r(s-t)} U(s, W_s, S_s, (\pi^1_s, \pi^2_s, \pi^3_s)) ds \right] \]

\[ = e^{-rdt} \sup_{(\pi^1, \pi^2, \pi^3) \in \Lambda_{t+dt}} E_t w_t W_t + dW_t S_t + ds_t \left[ \int_t^{t+dt} e^{-r(s-t)} U(s, W_s, S_s, (\pi^1_s, \pi^2_s, \pi^3_s)) ds \right] \]

\[ = e^{-rdt} U(t + dt, W_t + dW_t, S_t + dS_t). \tag{A.2.2} \]

Substitute (A.2.2) for (A.2.1),

\[ u(t, W_t, S_t) = \sup_{(\pi^1, \pi^2, \pi^3) \in \Lambda_{t+dt}} E_t w_t S_t \left[ \int_t^{t+dt} e^{-r(s-t)} U(s, W_s, S_s, (\pi^1_s, \pi^2_s, \pi^3_s)) ds \right. \]

\[ \left. + e^{-rdt} U(t + dt, W_t + dW_t, S_t + dS_t) \right] \], \tag{A.2.3} \]

Here, approximate \( e^k \approx 1 + k, \) then \( e^{-rdt} \approx 1 - rdt = 1. \)

Therefore, (A.2.3) can be rewritten

\[ u(t, W_t, S_t) = \sup_{(\pi^1, \pi^2, \pi^3) \in \Lambda_{t+dt}} E_t w_t S_t \left[ \int_t^{t+dt} e^{-r(s-t)} U(s, W_s, S_s, (\pi^1_s, \pi^2_s, \pi^3_s)) ds \right. \]

\[ \left. + (1 - 0) U(t + dt, W_t + dW_t, S_t + dS_t) \right] \], \tag{A.2.4} \]

Do Taylor expansion for the second term on the right side of (A.2.4), and apply Ito formula\(^{10} \) for it,

\[ U(t + dt, W_t, S_t, dS_t) \triangleq U(t + dt, W_t + dW_t S_t, S_t^2 + dS_t^2, S_t^3 + dS_t^3) \]

\[ = U(t, W_t, S_t, S_t^2, S_t^3) + \left[ \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial W_t} dW_t + \sum_{i=1}^{3} \frac{\partial U}{\partial S_t^i} dS_t^i + \frac{1}{2} \frac{\partial^2 U}{\partial W_t^2} dW_t \cdot dW_t + \sum_{i=1}^{3} \frac{\partial^2 U}{\partial S_t^i} dS_t^i \right] \]

\[ + \frac{\partial^2 U}{\partial S_t^2} dS_t^2 \cdot dS_t^2 + \frac{\partial^2 U}{\partial W_t^2} dW_t \cdot dS_t^2 + \frac{\partial^2 U}{\partial W_t \partial S_t^2} dW_t \cdot dS_t^2 \]

\[ + \frac{\partial^2 U}{\partial W_t \partial S_t^3} dW_t \cdot dS_t^3 \quad (i \neq j). \]

\(^{10} \) dB\( t \) dB\( t \) = dt, dB\( t \) dB\( t \) = dt \cdot dt = 0 (i \neq j).
\[
\begin{align*}
+ \frac{\partial^2 U}{\partial S^2} \cdot dS^2_1 \cdot dS^2_2 + \frac{\partial^2 U}{\partial S^2} \cdot dS^2_3 \cdot dS^2_1 + \frac{\partial^2 U}{\partial S^2} \cdot dS^2_3 \cdot dS^2_2 \right]. \tag{A.2.5}
\end{align*}
\]

Integrate the first term on the right side of (A.2.4), and get it substituted for (A.2.5), and then divide it by dt and dt→0, then

\[
0 = -u_t - \sup_{\pi_1,\pi_2,\pi_3} \left[ (\pi_1 (\mu_1 + \delta Z)s_1 + \pi_2 \mu_2 s_2 + \pi_3 \mu_3 s_3)u_w + (\mu_1 + \delta Z)s_1 u_s, \right. \\
+ \mu_2 s_3 u_s, \\
\mu s_3 u_s + \pi_1 \sigma^2_1 s_1 \cdot u_{s_1} + \pi_2 \sigma^2_2 s_2 \cdot u_{s_2} + \pi_3 \sigma^2_3 s_3 \cdot u_{s_3}, \\
+ \frac{1}{2} (\pi_1 \sigma^2_1 s_1 + \pi_2 \sigma^2_2 s_2 + \pi_3 \sigma^2_3 s_3)u_{ww} \\
\left. + \frac{1}{2} (\sigma^2_1 s_1^2 u_{s_1} + \sigma^2_2 s_2^2 u_{s_2} + \sigma^2_3 s_3^2 u_{s_3}) \right]. \tag{A.2.6}
\]

2.2 Ansatz

Set terminal condition as \( u(T, w, s_1, s_2, s_3) = U(w) = -\exp(-\gamma w) \), which is completed under the \( 0 \leq t < T, w, 0 \leq s_1, 0 \leq s_2, 0 \leq s_3 \). Now, let it transform as follows.

Suppose,

\[
\begin{align*}
s_1 &= e^{s_1}, s_2 = e^{s_2}, s_3 = e^{s_3}, u(t, w, s_1, s_2, s_3) &= -e^{-\gamma w} g(t, x, y, z), \quad \text{then} \quad u_t &= -e^{-\gamma w} g_t, \quad u_w = -e^{-\gamma w} g_w, \quad (\gamma) = ye^{-\gamma w} g, \\
u_{s_1} &= -\frac{s_1 e^{-\gamma w} g_{s_1}}{s_1} + e^{-\gamma w} g_{s_1}, \\
u_{s_2} &= \frac{s_2 e^{-\gamma w} g_{s_2}}{s_2} + e^{-\gamma w} g_{s_2}, \\
u_{s_3} &= \frac{s_3 e^{-\gamma w} g_{s_3}}{s_3} + e^{-\gamma w} g_{s_3}, \\
u_{w_s} &= ye^{-\gamma w} g_x, \quad u_{w_s} = ye^{-\gamma w} g_y, \quad u_{w_s} = ye^{-\gamma w} g_z.
\end{align*}
\]

Substitute these calculations for (A.2.6), and divide by the \( e^{-\gamma w} \), the following relation can be derived.

\[
0 = -g_t + \sup_{\pi_1,\pi_2,\pi_3} \left[ (\pi_1 (\mu_1 + \delta Z)s_1 + \pi_2 \mu_2 s_2 + \pi_3 \mu_3 s_3)g_y - (\mu_1 + \delta Z)g_x - \mu_2 g_y, \\
- \pi_2 g_x + \pi_1 \sigma^2_1 s_1 g_{s_1} + \pi_2 \sigma^2_2 s_2 g_{s_2} + \pi_3 \sigma^2_3 s_3 g_{s_3}, \\
\frac{1}{2} \pi_1 \sigma^2_1 s_1^2 \gamma^2 g - \frac{1}{2} \pi_2 \sigma^2_2 s_2^2 \gamma^2 g - \frac{1}{2} \pi_3 \sigma^2_3 s_3^2 \gamma^2 g, \\
\frac{1}{2} \sigma^2_1 (g_{xx} - g_x) + \frac{1}{2} \sigma^2_2 (g_{yy} - g_y), \\
\frac{1}{2} \sigma^2_3 (g_{zz} - g_z). \tag{A.2.7}
\right]
\]
Where terminal condition is $g(T, x, y, z) = 1$. Focus on the Sup term of (A.2.7) 
$$
\sup_{\pi_1, \pi_2, \pi_3} \left\{ \pi_1 (\mu_1 + \delta Z) s_1 + \pi_2 \mu_2 s_2 + \pi_3 \mu_3 s_3 \right\} y g - (\mu_1 + \delta Z) g y - \mu_2 g y
$$
\[ - \mu_3 g z + \pi_1 \sigma_1^2 s_1 y g_x + \pi_2 \sigma_2^2 s_2 y g_y + \pi_3 \sigma_3^2 s_3 y g_z \]
\[ - \frac{1}{2} \pi_1^2 \sigma_1^2 s_1 y^2 g - \frac{1}{2} \pi_2^2 \sigma_2^2 s_2 y^2 g - \frac{1}{2} \pi_3^2 \sigma_3^2 s_3 y^2 g \]
\[ \frac{1}{2} \pi_1^2 (g_{xx} - g_x) - \frac{1}{2} \pi_2^2 (g_{yy} - g_y) - \frac{1}{2} \pi_3^2 (g_{zz} - g_z) = f , \]

Then,
\[ \pi_1^* = \frac{\partial f}{\partial \pi_1} = \frac{(\mu_1 + \delta Z) g + \sigma_1^2 g_x}{\sigma_1^2 s_1 y g} , \quad (A.2.8) \]
\[ \pi_2^* = \frac{\partial f}{\partial \pi_2} = \frac{\mu_2 g + \sigma_2^2 g_y}{\sigma_2^2 s_2 y g} , \quad (A.2.9) \]
\[ \pi_3^* = \frac{\partial f}{\partial \pi_3} = \frac{\mu_3 g + \sigma_3^2 g_z}{\sigma_3^2 s_3 y g} . \quad (A.2.10) \]

Substitute (A.2.8), (A.2.9) and (A.2.10) for the (A.2.7), then the following relation can be derived.
\[ g_t = \frac{1}{2} \left( \frac{(\mu_1 + \delta Z)^2}{\pi_1^2} \left( \frac{\mu_2^2}{\sigma_2^2} + \frac{\mu_3^2}{\sigma_3^2} \right) g + \frac{\sigma_1^2}{2} g_x + \frac{\sigma_2^2}{2} g_y + \frac{\sigma_3^2}{2} g_z \right) + \frac{1}{2} \left( \pi_1^2 \sigma_1^2 g_x + \pi_2^2 \sigma_2^2 g_y + \pi_3^2 \sigma_3^2 g_z \right) - \frac{1}{2} \left( \sigma_1^2 g_{xx} + \sigma_2^2 g_{yy} + \sigma_3^2 g_{zz} \right) . \quad (A.2.11) \]

2.3 Call-Hop transformation

To eliminate no linearity (A.2.11), transform using with $\Phi(t, X) = -\log (g(t, x, y, z))$.

Let’s define $X, g(\cdot, \cdot)$ as $X = \mu_1 + \delta (a + x + \beta_1 y + \beta_2 z)$ , $g(t, x, y, z) = -e^{\Phi(t,X)}$, then
\[ \Phi_t = - \frac{1}{2} \left( \frac{X^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} + \frac{\mu_3^2}{\sigma_3^2} \right) + \frac{1}{2} \left( \sigma_1^2 + \beta_1 \sigma_2^2 + \beta_2 \sigma_3^2 \right) g(\Phi_X) \]
\[ - \frac{1}{2} \left( \sigma_1^2 + \beta_1 \sigma_2^2 + \beta_2 \sigma_3^2 \right) (\delta^2 \Phi_{xx}) . \quad (A.2.12) \]

(A.2.12) is Riccaci type partial differential equation, therefore $\Phi(t,X) = a(t)X^2 + b(t)X + c(t)$ is the solution of (A.2.12). This one satisfies the terminal condition under $\Phi(T, X) = 0$ under $X \in \mathbb{R}$, $0 \le t < T$.

2.4 Closed form solution for three assets type

Substitute $\Phi_t = a(t)X^2 + b(t)X + c(t)$, $\Phi_X = 2a(t)X + b(t)$, $\Phi_{XX} = 2a(t)$ for (A.2.12)$^\dagger$.

$^\dagger$ The process of transformation to get to Riccaci type partial differential equation, t is called Call hop transformation.
\[ a(t)'X^2 + b(t)'X + c(t)' = -\frac{1}{2} \left( \frac{X^2}{\sigma_1^2} + \frac{\mu_1^2}{\sigma_2^2} + \frac{\mu_2^2}{\sigma_3^2} \right) + \frac{1}{2} \delta(\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2)(2a(t)X + b(t)) \]

\[ = -\frac{1}{2} \delta^2(\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2)(2a(t)) , \quad (A.2.13) \]

Comparing the coefficients of $X^2, X, 0$. It is the ordinary differential equation which type of separation variables.

Considering the terminal condition to get the following equations.

\[ a(t) = \frac{1}{2} \frac{(T-t)}{\sigma_1^2}, \quad (A.2.14) \]

\[ b(t) = -\frac{1}{4} \left( \frac{\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2}{\sigma_1^2} \delta \right)(T-t)^2. \quad (A.2.15) \]

Paying attention to $\Phi(t, X) = a(t)X^2 + b(t)X + c(t), \Phi_x = 2a(t)X + b(t)$, from (A.2.8) to (A.2.10), the optimal trading amount can be calculated as follows.

\[ \pi_i^* \triangleq \left( \mu_i + \delta Z \right) g + \sigma_i^2 g_x \equiv \frac{(X \cdot -e^\Phi) + (\sigma_i^2 \cdot \Phi \cdot \delta \cdot e^\Phi)}{\sigma_i^2 s_1 \gamma (-e^\Phi)} \]

\[ = \frac{(X \cdot -e^\Phi) + (\sigma_i^2 \cdot -(2a(t)X + b(t)) \cdot \delta \cdot e^\Phi)}{\sigma_i^2 s_1 \gamma (-e^\Phi)} \]

\[ = \frac{X}{s_1 \sigma_i^2 \gamma} + \frac{\delta(-2a(t)X - b(t))}{s_1 \gamma} . \quad (A.2.16) \]

Substitute (A.2.14) and (A.2.15) for (A.2.16),

\[ \pi_1^* = \frac{\left( \mu_1 + \delta Z \right)}{s_1 \gamma \sigma_1^2} - \delta \frac{\left( \mu_1 + \delta Z \right)}{s_1 \gamma \sigma_1^2} (T-t) + \frac{1}{4} \frac{\delta^2(\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2)}{s_1 \gamma \sigma_1^2} (T-t)^2, \]

The optimal trading amount of $S_1$ can be derived and also for others assets.

Summarize,

\[ \pi_1^* = \frac{\mu_1 + \delta Z}{s_1 \gamma \sigma_1^2} - \delta \frac{\mu_1 + \delta Z}{s_1 \gamma \sigma_1^2} (T-t) + \frac{1}{4} \frac{\delta^2(\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2)}{s_1 \gamma \sigma_1^2} (T-t)^2, \]

\[ \pi_2^* = \frac{\mu_2}{s_2 \gamma \sigma_2^2} - \beta_2 \delta \frac{\mu_1 + \delta Z}{s_2 \gamma \sigma_2^2} (T-t) + \frac{1}{4} \frac{\delta^2(\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2)}{s_2 \gamma \sigma_2^2} (T-t)^2, \]

\[ \pi_3^* = \frac{\mu_3}{s_3 \gamma \sigma_3^2} - \beta_3 \delta \frac{\mu_1 + \delta Z}{s_3 \gamma \sigma_3^2} (T-t) + \frac{1}{4} \frac{\delta^2(\sigma_1^2 + \beta_1\sigma_2^2 + \beta_2\sigma_3^2)}{s_3 \gamma \sigma_3^2} (T-t)^2. \]
References


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